Announcements

1) Amanda has more mentoring hours
M 9:30-11 2090 CB

Proposition. Suppose $\boxed{a_n \geq 0}$ $\forall n \in \mathbb{N}$

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if

$(S_k)_{k=1}^{\infty}$ is bounded.

proof. $\Rightarrow$) Suppose $\sum_{n=1}^{\infty} a_n$

converges. Then this means

$$\lim_{k \to \infty} \sum_{n=1}^{k} a_n = \lim_{k \to \infty} S_k \text{ exists,}$$

Call the limit L.

Then $\forall\ \varepsilon > 0$, $\exists\ K \in \mathbb{N}$ such that

$$|S_k - L| < \varepsilon \quad \forall\ k \geq K.$$

Then $\forall\ k \geq K$,

$$\varepsilon > |S_k - L| \geq \big||S_k| - |L|\big|$$

$$\Rightarrow \quad |S_k| < |L| + \varepsilon \quad \forall\ k \geq K.$$

Then if

$$M = \max\{|S_1|, |S_2|, \ldots, |S_{K-1}|, |L| + \varepsilon\},$$

$$|S_k| \leq M \quad \forall\ k \in \mathbb{N}$$

$\Leftarrow$ Suppose $(S_k)_{k\in\mathbb{N}}$ is bounded.

Since $a_n \geq 0 \ \forall n \in \mathbb{N}$,

$$S_k = \sum_{n=1}^{k} a_n \leq \sum_{n=1}^{k} a_n + a_{k+1} = S_{k+1}$$

since $a_{k+1} \geq 0$.

Then $(S_k)_{k\in\mathbb{N}}$ is bounded and monotonically increasing, so converges by monotone Convergence Theorem. Hence, $\sum_{n=1}^{\infty} a_n$ converges $\square$

Remarks

1) $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow (S_k)_{k\in\mathbb{N}}$ bounded is true for any sequence $(a_n)_{n\in\mathbb{N}}$, not just non-negative sequences.

2) The converse if $(a_n)_{n\in\mathbb{N}}$ is not non-negative is false:

$a_n = (-1)^{n+1}$ Then $0 \le S_k \le 1$, but does not converge

Theorem: (Cauchy condensation)

$(a_n)_{n\in\mathbb{N}}$ monotonically decreasing and $\boxed{a_n \geq 0}$ $\forall\, n \in \mathbb{N}$.

Then $\sum_{n=1}^{\infty} a_n$ converges

if and only if

$\sum_{k=0}^{\infty} (2^k a_{2^k})$ converges

proof. $\Rightarrow$ Suppose $\sum_{n=1}^{\infty} a_n$

converges

Want to conclude

$$\sum_{k=0}^{\infty} \left(2^k a_{2^k}\right) \quad \text{converges.}$$

$$\sum_{n=1}^{2^m} a_n = a_1 + a_2 + a_3 + a_4 + - + a_{2^m}$$

$$\sum_{k=0}^{t} \left(2^k a_{2^k}\right) = a_1 + 2a_2 + 4a_4 + — + 2^t a_{2^t}$$

$$= a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4)$$

$$+ — + 2^t a_{2^t}$$

$$\leq a_1 + (a_1 + a_2)$$

$$+ (a_3 + a_4 + a_3 + a_4)$$

the next term is

$$8a_8$$

$$= (a_8 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8)$$

$$\leq (a_5 + a_5 + a_6 + a_6 + a_7 + a_7 + a_8 + a_8)$$

$$= 2(a_5 + a_6 + a_7 + a_8).$$

(since $a_5 \geq a_6 \geq a_7 \geq a_8$)

In general, one can show by induction that

$$2^k a_{2^k} \leq 2 \sum_{n=2^{k-1}+1}^{2^k} a_n$$

($a_n \geq a_{n-1} \geq \cdots$)

Then

$$\sum_{k=0}^{t} \left(2^k a_{2^k}\right) \leq 2 \sum_{n=1}^{2^t} a_n$$

$$\leq 2 \sum_{n=1}^{\infty} a_n = 2L$$

Since $\sum_{n=1}^{\infty} a_n$ converges.

Then with $b_{k+1} = 2^k a_{2^k}$,

the partial sums $(S_t)_{t \in \mathbb{N}}$ of the series $\sum b_k$ are bounded and $a_n \geq 0 \ \forall n \in \mathbb{N} \Rightarrow b_k \geq 0 \ \forall k \in \mathbb{N}$.

So by the previous proposition,

$$\sum_{k=1}^{\infty} b_k = \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges}$$

$\Leftarrow$ Suppose $\displaystyle\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Then $\displaystyle\sum_{n=1}^{2^k-1} a_n =$

$$a_1 + \underbrace{a_2 + a_3}_{\le a_2 + a_2} + \underbrace{a_4 + a_5 + a_6 + a_7}_{\le a_4 + a_4 + a_4 + a_4} + \dots + a_{2^k-1}$$

(monotone decreasing)

$$\le a_1 + 2a_2 + 4a_4 + \dots + 2^{k-1} a_{2^{k-1}}$$

So $$\sum_{n=1}^{2^k-1} a_n \leq \sum_{m=0}^{2^{k-1}} 2^m a_{2^m}$$

$$\leq \sum_{m=0}^{\infty} 2^m a_{2^m} = L$$

since $\sum_{m=0}^{\infty} 2^m a_{2^m}$ converges.

Since $a_n \geq 0$ $\forall n$, then $\forall t \in \mathbb{N}$, $\exists$ $k \in \mathbb{N}$ with

$$\sum_{n=1}^{t} a_n \leq \sum_{n=1}^{2^k-1} a_n \leq L.$$

Again invoking the previous proposition, since

$(S_k)_{k \in \mathbb{N}}$ is bounded for

$S_k = \sum_{n=1}^{k} a_n$ and $a_n \geq 0 \ \forall n \in \mathbb{N}$,

$\sum_{n=1}^{\infty} a_n$ converges. $\square$.

Consequences: Consider $p \in \mathbb{R}$

and examine $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Apply Cauchy Condensation Theorem:

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if

$$\sum_{k=0}^{\infty} 2^k\left(\frac{1}{(2^k)^p}\right) = \sum_{k=0}^{\infty}\left(\frac{1}{2^{p-1}}\right)^k$$

converges - a geometric series!

The latter series converges

whenever

$$\frac{1}{2^{p-1}} < 1 \iff p > 1.$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ Converges if $p > 1$,

diverges otherwise.

What about if we delete hypotheses?

Then False.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{converges}$$

(will show next class) but

$$\sum_{k=0}^{\infty} 2^k \frac{(-1)^{2^k}}{2^k} = -1 + \sum_{k=1}^{\infty} 1$$

diverges

Note: $a_n = \frac{(-1)^n}{n}$, not non-negative, doesn't decrease